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Generalized vector coherent state theory of sp(2N, $\mathbb{R}$) vector operators and of sp(2N, $\mathbb{R}$) $\supset$ u(N) reduced Wigner coefficients

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Abstract. sp(2N, $\mathbb{R}$) $\supset$ u(N) reduced Wigner coefficients coupling a positive discrete series unitary irreducible representation with the 2N-dimensional non-unitary irreducible representation characterized by $\langle 1 \rangle$ are evaluated by making use of generalized vector coherent state techniques General formulae are given in terms of the K-matrix elements of vector coherent state theory and of some u(N) 6j- or 9j-type recoupling coefficients. Explicit analytic expressions are obtained in some important special cases Such results should allow a recent determination of wsp(2N, $\mathbb{R}$) $=$ w(N) $\oplus$ sp(2N, $\mathbb{R}$) and osp(P/2N, $\mathbb{R}$) matrix representations to be completed, because both the w(N) generators and the osp(P/2N, $\mathbb{R}$) odd generators transform under the sp(2N, $\mathbb{R}$) irrep $\langle 1 \rangle$

1. Introduction

The non-compact symplectic algebra sp(2N, $\mathbb{R}$) being the main component of the N-dimensional harmonic oscillator dynamical algebra (Hwa and Nuyts 1966, Moshinsky and Quesne 1971, Wybourne 1974) appears in many physical problems. The harmonic oscillator eigenstates carry infinite-dimensional unitary irreducible representations (irreps) of sp(2N, $\mathbb{R}$) belonging to the positive discrete series. However physically relevant operators usually transform under some finite-dimensional non-unitary irrep. To exploit the Wigner–Eckart theorem with respect to sp(2N, $\mathbb{R}$), one therefore needs the sp(2N, $\mathbb{R}$) Wigner coefficients coupling a positive discrete series unitary irrep with a finite-dimensional non-unitary one to give another positive discrete series unitary irrep. The latter however remain unknown, except for the case where $N = 1$, for which they coincide with some su(1, 1) Wigner coefficients determined many years ago by Ui (1968).

Finite-dimensional non-unitary irreps also play an important role whenever sp(2N, $\mathbb{R}$) is embedded into some larger finite-dimensional algebra g. In such a case, they indeed govern the transformation properties of the additional generators under sp(2N, $\mathbb{R}$), so that a complete determination of the matrix irreps of g in a $g \supset$ sp(2N, $\mathbb{R}$) $\supset$ u(N) basis requires an explicit knowledge of the previously mentioned sp(2N, $\mathbb{R}$) Wigner coefficients. Two examples of such embeddings were recently studied in detail, corresponding to $g =$ wsp(2N, $\mathbb{R}$) $\equiv$ w(N) $\oplus$ sp(2N, $\mathbb{R}$) (Quesne 1990a) and $g =$ osp(P/2N, $\mathbb{R}$), $P = 2M$ or $2M + 1$ (Quesne 1990b), respectively. Here w(N) denotes the Heisenberg–Weyl algebra generated by N pairs of boson creation and annihilation

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operators (and the unit operator), and $\mathrm{osp}(P/2N,\mathbb{R})$ the orthosymplectic superalgebra whose even subalgebra is $\mathrm{so}(P)\oplus\mathrm{sp}(2N,\mathbb{R})$ In both cases, the additional generators are vector operators with respect to $\mathrm{sp}(2N,\mathbb{R})$, i.e. they transform under the $2N$-dimensional irrep $\langle 1\rangle$.

The aim of the present paper is to determine the $\mathrm{sp}(2N,\mathbb{R})$ Wigner coefficients which couple a positive discrete series irrep with the irrep $\langle 1\rangle$. For such a purpose, we shall make use of the vector coherent state (VCS) theory (Deenen and Quesne 1984, Rowe 1984) and its associated K-matrix technique (Rowe 1984, Castaños *et al* 1985, Deenen and Quesne 1985, Hecht 1987, Rowe *et al* 1988). VCS theory is indeed not only useful for determining explicit matrix representations of many Lie algebras and superalgebras, but it is also applicable to the Wigner–Racah calculus of higher rank algebras (Hecht *et al* 1987, Le Blanc and Hecht 1987, Le Blanc and Biedenharn 1989). Very recently, it has been generalized to include the VCS realizations of simple operators lying outside the algebras, thereby providing a very powerful tool for a detailed evaluation of their matrix elements and thence of Wigner coefficients (Hecht 1989, Hecht and Biedenharn 1990, Hecht and Chen 1990).

So far, the generalized VCS theory has been applied to the calculation of $g\supset h$ reduced Wigner coefficients, for which the reduction of a product of g (resp. h) irreps into a direct sum of g (resp. h) irreps was known beforehand, namely $\mathrm{u}(N)\supset\mathrm{u}(N-1)$ (Hecht *et al* 1987, Le Blanc and Hecht 1987, Le Blanc and Biedenharn 1989, Hecht and Biedenharn 1990), $\mathrm{sp}(4)\simeq\mathrm{so}(5)\supset\mathrm{u}(2)$ (Hecht 1989), and $\mathrm{sp}(6)\supset\mathrm{u}(3)$ reduced Wigner coefficients (Hecht and Chen 1990). In the present paper, we are confronted with a more challenging case. except when $N=1$, no information is indeed available about the decomposition of the product of a positive discrete series unitary irrep with a vector irrep of $\mathrm{sp}(2N,\mathbb{R})$. However, the generalized VCS technique will prove a very powerful tool for determining such a branching rule. As in its previous applications, it can be used to express the $\mathrm{sp}(2N,\mathbb{R})\supset\mathrm{u}(N)$ reduced Wigner coefficients in terms of the known K-matrix and of some $\mathrm{u}(N)$ $6j$- or $9j$-type recoupling coefficients. In various important cases, very explicit analytic expressions are therefore obtained for these coefficients

In the following section of this paper, the $\mathrm{sp}(2N,\mathbb{R})\supset\mathrm{u}(N)$ VCS theory is briefly reviewed. Then, in section 3, the generalized theory is applied to evaluating the matrix elements of boson operators. Finally, in section 4, general formulae for the corresponding $\mathrm{sp}(2N,\mathbb{R})\supset\mathrm{u}(N)$ reduced Wigner coefficients are determined, and some simple examples are worked out in detail.

2. Vector coherent state theory of $\mathrm{sp}(2N,\mathbb{R})\supset\mathrm{u}(N)$

The $\mathrm{sp}(2N,\mathbb{R})$ algebra is spanned by the $\mathrm{u}(N)$ subalgebra generators $E_{ij}=(E_{ji})^{\dagger}$, i, $j=1,\ldots,N$, a set of commuting raising operators $D^{\dagger}_{ij}=D^{\dagger}_{ji}$, $i,j=1,\ldots,N$, and the corresponding Hermitian conjugate lowering operators $D_{ij}=D_{ji}=(D^{\dagger}_{ij})^{\dagger}$, $i,j=1,\ldots,N$ (Deenen and Quesne 1984). Such operators, whose non-vanishing commutators are given by

$$\begin{aligned}
&[E_{ij},E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}\\
&[E_{ij},D^{\dagger}_{kl}]=\delta_{jk}D^{\dagger}_{il}+\delta_{jl}D^{\dagger}_{ik}\qquad [E_{ij},D_{kl}]=-\delta_{ik}D_{jl}-\delta_{il}D_{jk}\qquad (2.1)\\
&[D_{ij},D^{\dagger}_{kl}]=\delta_{ik}E_{lj}+\delta_{il}E_{kj}+\delta_{jk}E_{li}+\delta_{jl}E_{ki}
\end{aligned}$$

can be realized in terms of Nn pairs of boson creation and annihilation operators $a^\dagger_{is}$, a_{is}, $i = 1, \ldots, N$, $s = 1, \ldots, n$, as follows.

$$D^\dagger_{ij} = \sum_{s=1}^{n} a^\dagger_{is} a^\dagger_{js} \qquad D_{ij} = \sum_{s=1}^{n} a_{is} a_{js} \qquad E_{ij} = \sum_{s=1}^{n} a^\dagger_{is} a_{js} + \tfrac{1}{2} n \delta_{ij}. \tag{2.2}$$

The sp(2N, $\mathbb{R}$) positive discrete series irreps can be induced from a lowest-weight u(N) irrep $\{\omega\} = \{\omega_1 \omega_2 \ldots \omega_N\}$ and are denoted by $\langle\omega\rangle = \langle\omega_1\omega_2 \ldots \omega_N\rangle$. Here ω_i, $i = 1, \ldots, N$, are some integers or half-integers subject to the conditions $\omega_1 \geqslant \omega_2 \geqslant \ldots \geqslant \omega_N > N$ (implying that, in the realization (2.2), n must satisfy the condition $n \geqslant 2N$). A non-orthonormal basis of their carrier space consists of the state vectors

$$|\langle\omega\rangle\{\nu\}\rho\{h\}\chi\rangle = [P^{\{\nu\}}(\mathbf{D}^\dagger) \times |\{\omega\}\rangle]^{\rho\{h\}}_{\chi} \tag{2.3}$$

where $|\{\omega\}\alpha\rangle$ is an orthonormal basis of the lowest-weight u(N) irrep carrier space or 'intrinsic' subspace, and $P^{\{\nu\}}(\mathbf{D}^\dagger)$ is a polynomial tensor transforming under the u(N) irrep $\{\nu\} = \{\nu_1\nu_2 \ldots \nu_N\}$, where the ν_i, $i = 1, \ldots, N$, are some non-negative even integers (Deenen and Quesne 1982, Le Blanc and Rowe 1987). The square bracket denotes the u(N) coupling $\{\omega\} \times \{\nu\} \to \{h\}$ to states of resultant u(N) symmetry specified by $\{h\} = \{h_1 h_2 \ldots h_N\}$ (note that throughout this paper all couplings are assumed to be ordered sequentially from right to left). The multiplicity label ρ is needed if $\{h\}$ is contained more than once in $\{\omega\} \times \{\nu\}$. The label χ can be any convenient set of labels characterizing the row of $\{h\}$.

The sp(2N, $\mathbb{R}$) VCS (also called partially coherent states) are parametrized by $\frac{1}{2}N(N+1)$ complex variables $z_{ij} = z_{ji}$, $i, j = 1, \ldots, N$, and by the discrete index α labelling the basis of the intrinsic subspace (Deenen and Quesne 1984, Rowe 1984, Hecht 1987). They are defined by

$$|\mathbf{z}, \alpha\rangle = \exp(\tfrac{1}{2} \operatorname{tr} \mathbf{z}^* \mathbf{D}^\dagger) |\{\omega\}\alpha\rangle \tag{2.4}$$

where $\operatorname{tr} \mathbf{z}^*\mathbf{D}^\dagger = \Sigma_{ij} z^*_{ij} D^\dagger_{ji}$ denotes the trace of the $N \times N$ matrix $\mathbf{z}^*\mathbf{D}^\dagger$. In the VCS representation, state vectors $|\Psi\rangle$ are represented by *holomorphic functions* $\Psi(\mathbf{z})$, *taking* vector values in the intrinsic subspace

$$\Psi_\alpha(\mathbf{z}) = \langle \mathbf{z}, \alpha|\Psi\rangle = \langle\{\omega\}\alpha| \exp(\tfrac{1}{2} \operatorname{tr} \mathbf{z}\mathbf{D})|\Psi\rangle. \tag{2.5}$$

Operators O are represented by $\mathbf{z}$-space operators $\Gamma(O)$, taking matrix values in the intrinsic subspace

$$[\Gamma(O)\Psi(\mathbf{z})]_\alpha = \sum_{\alpha'} \Gamma_{\alpha\alpha'}(O)\Psi_{\alpha'}(\mathbf{z}) \tag{2.6}$$

and defined by

$$\begin{aligned} [\Gamma(O)\Psi(\mathbf{z})]_\alpha &= \langle\{\omega\}\alpha| \exp(\tfrac{1}{2} \operatorname{tr} \mathbf{z}\mathbf{D}) O |\Psi\rangle \\ &= \langle\{\omega\}\alpha| \{O + [\tfrac{1}{2} \operatorname{tr} \mathbf{z}\mathbf{D}, O] + \tfrac{1}{2}[\tfrac{1}{2} \operatorname{tr} \mathbf{z}\mathbf{D}, [\tfrac{1}{2} \operatorname{tr} \mathbf{z}\mathbf{D}, O]] + \ldots\} \\ &\quad \times \exp(\tfrac{1}{2} \operatorname{tr} \mathbf{z}\mathbf{D})|\Psi\rangle. \end{aligned} \tag{2.7}$$

The VCS representation of the sp(2N, $\mathbb{R}$) generators was obtained by Deenen and Quesne (1984) and Rowe (1984). It can be expressed in terms of the variables z_{ij}, the corresponding differential operators $\nabla_{ij} = (1 + \delta_{ij})\, \partial/\partial z_{ij}$, and the intrinsic u($N$) generators $\mathbb{E}_{ij}$. The latter, defined by

$$[\mathbb{E}_{ij}\Psi(\mathbf{z})]_\alpha = \sum_{\alpha'} \langle\{\omega\}\alpha|E_{ij}|\{\omega\}\alpha'\rangle \Psi_{\alpha'}(\mathbf{z}) \tag{2.8}$$

commute with z_{ij} and ∇_{ij}, and only act on the intrinsic states (this means that, when calculating matrix elements, they must be worked through to the left so that they can act on the intrinsic bra). In a matrix notation, wherein $\mathbf{z}$, $\boldsymbol{\nabla}$, $\mathbb{E}$ and $\mathscr{E}$ are $N \times N$ matrices and $\tilde{\mathbb{E}}$ is the transpose of $\mathbb{E}$, one can write

$$\begin{aligned} &\Gamma(\mathbf{D}) = \boldsymbol{\nabla} \\ &\Gamma(\mathbf{E}) = \mathbb{E} + \mathscr{E} \qquad \mathscr{E} = \mathbf{z}\boldsymbol{\nabla} \\ &\Gamma(\mathbf{D}^{\dagger}) = \mathbf{z}\tilde{\mathbb{E}} + \mathbb{E}\mathbf{z} + (\mathbf{z}\boldsymbol{\nabla} - N - 1)\mathbf{z} = [\Lambda, \mathbf{z}] \end{aligned} \tag{2.9}$$

where

$$\Lambda = \tfrac{1}{2}\,\mathrm{tr}[(\mathbb{E} + \mathscr{E})^2 - \tfrac{1}{2}\mathscr{E}^2 - \tfrac{1}{2}(N+1)\mathscr{E}]. \tag{2.10}$$

Instead of endowing the $\mathbf{z}$-space with the vcs scalar product, with respect to which $\Gamma(\mathbf{D}^{\dagger})$ would be the adjoint of $\Gamma(\mathbf{D})$, it is advantageous to equip it with a vector-Bargmann scalar product (Bargmann 1961). With respect to the latter, $\boldsymbol{\nabla}$ is the adjoint of $\mathbf{z}$, so that a $\mathbf{z}$-space orthonormal basis is easily constructed (Rowe 1984, Deenen and Quesne 1985, Hecht 1987) and given by

$$|\langle\omega\rangle\{\nu\}\rho\{h\}\chi) = [P^{\{\nu\}}(\mathbf{z}) \times |\{\omega\}\rangle]^{\rho\{h\}}_{\chi} \tag{2.11}$$

where $P^{\{\nu\}}(\mathbf{z})$ is obtained from the polynomial tensor $P^{\{\nu\}}(\mathbf{D}^{\dagger})$ of (2.3) by substituting $\mathbf{z}$ for $\mathbf{D}^{\dagger}$. To denote the vector-Bargmann basis functions, round parentheses are used instead of the angular carets of (2.3), which designate standard Hilbert space vectors. Throughout this paper, the appearance of round parentheses in matrix elements will mean that they are calculated by $\mathbf{z}$-space integrations with the Bargmann measure and pure intrinsic space operations, whereas the presence of angular carets will signal standard Hilbert space operations.

The price paid for the existence of the simple $\mathbf{z}$-space orthonormal basis (2.11) is that the Γ representation (2.9) is a non-unitary realization of the generators O. It can, however, be transformed into a unitary realization

$$\gamma(O) = K^{-1}\Gamma(O)K \tag{2.12}$$

via a similarity transformation with an operator K (Deenen and Quesne 1982, 1985, Rowe 1984, Castaños *et al* 1985, Hecht 1987). Since $\Gamma(\mathbf{E})$ is already Hermitian with respect to the vector-Bargmann scalar product, K may be chosen invariant under u(N) so that its matrix elements in the basis (2.11) are diagonal in $\{h\}$ and independent of χ. The condition $\gamma(\mathbf{D}^{\dagger}) = (\gamma(\mathbf{D}))^{\dagger}$ then leads to the equation

$$[\Lambda, \mathbf{z}]KK^{\dagger} = KK^{\dagger}\mathbf{z} \tag{2.13}$$

from which the matrix elements of $KK^{\dagger}$,

$$(KK^{\dagger}(\{\omega\},\{h\}))_{\{\nu'\}\rho',\{\nu\}\rho} = (\langle\omega\rangle\{\nu'\}\rho'\{h\}\chi|KK^{\dagger}|\langle\omega\rangle\{\nu\}\rho\{h\}\chi) \tag{2.14}$$

can be recursively determined from $KK^{\dagger}(\{\omega\},\{\omega\}) \equiv 1$, as reviewed in the appendix. Some examples of analytic solutions for $KK^{\dagger}(\{\omega\},\{h\})$ are also given there.

Via some unitary matrix $U(\{\omega\},\{h\})$, every submatrix $KK^{\dagger}(\{\omega\},\{h\})$ can be converted to diagonal form $D(\{\omega\},\{h\})$, given by

$$UKK^{\dagger}U^{\dagger} = D = \mathrm{diag}(d_1, d_2, \ldots, d_R) \tag{2.15}$$

where all the eigenvalues d_r, $r=1,2,\ldots,R$, are non-vanishing (Rowe *et al* 1985). The eigenstates of $KK^\dagger(\{\omega\},\{h\})$ can be labelled by index r and written as

$$|\langle\omega\rangle\{h\}\chi r) = \sum_{\{\nu\}\rho} |\langle\omega\rangle\{\nu\}\rho\{h\}\chi)(U^\dagger)_{\{\nu\}\rho,r}. \tag{2.16}$$

From (2.15), one obtains

$$\begin{aligned} &K_{\{\nu\}\rho,r} = (d_r)^{1/2}(U^\dagger)_{\{\nu\}\rho,r} \qquad (K^\dagger)_{r,\{\nu\}\rho} = (d_r)^{1/2}U_{r,\{\nu\}\rho} \\ &(K^{-1})_{r,\{\nu\}\rho} = (d_r)^{-1/2}U_{r,\{\nu\}\rho} \qquad ((K^\dagger)^{-1})_{\{\nu\}\rho,r} = (d_r)^{-1/2}(U^\dagger)_{\{\nu\}\rho,r}. \end{aligned} \tag{2.17}$$

Whenever the submatrix $KK^\dagger(\{\omega\},\{h\})$ is one-dimensional, one has $U(\{\omega\},\{h\}) = I$ so that $K^\dagger(\{\omega\},\{h\}) = K(\{\omega\},\{h\})$.

The $KK^\dagger$ matrix can be interpreted as the overlap matrix of the non-orthonormal Hilbert space basis (2.3) (Deenen and Quesne 1985, Hecht 1987),

$$\langle\langle\omega\rangle\{\nu'\}\rho'\{h'\}\chi'|\langle\omega\rangle\{\nu\}\rho\{h\}\chi\rangle = \delta_{\{h'\},\{h\}}\delta_{\chi',\chi}(KK^\dagger(\{\omega\},\{h\}))_{\{\nu'\}\rho',\{\nu\}\rho}. \tag{2.18}$$

A Hilbert space orthonormal basis, corresponding to the vector-Bargmann orthonormal basis (2.16), is given by

$$|\langle\omega\rangle\{h\}\chi r\rangle = \sum_{\{\nu\}\rho} |\langle\omega\rangle\{\nu\}\rho\{h\}\chi\rangle((K^\dagger(\{\omega\},\{h\}))^{-1})_{\{\nu\}\rho,r}. \tag{2.19}$$

The matrix elements of an operator O in the orthonormal basis (2.19) can be evaluated in **z**-space by using the vector-Bargmann orthonormal basis (2.16) and the unitary realization $\gamma(O)$ of the operator. In the case where O transforms irreducibly under u(N), its u(N) reduced matrix elements can be written as

$$\begin{aligned} &\langle\langle\omega'\rangle\{h'\}r'\|O\|\langle\omega\rangle\{h\}r\rangle \\ &\quad = (\langle\omega'\rangle\{h'\}r'\|\gamma(O)\|\langle\omega\rangle\{h\}r) \\ &\quad = \sum_{\{\nu\}\rho}\sum_{\{\nu'\}\rho'} (K^{-1}(\{\omega'\},\{h'\}))_{r',\{\nu'\}\rho'}(\langle\omega'\rangle\{\nu'\}\rho'\{h'\}\|\Gamma(O)\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ &\quad\quad \times (K(\{\omega\},\{h\}))_{\{\nu\}\rho,r} \end{aligned} \tag{2.20}$$

where we have used (2.12). In writing (2.20), we have assumed that the u(N) coupling is multiplicity free; should it be otherwise, the reduced matrix elements would be labelled by some extra index. The interest in (2.20) is that the reduced matrix elements of O are expressed in terms of the known matrix elements of K and K^{-1}, and those of $\Gamma(O)$, which can be determined from some simple intrinsic space matrix elements and some **z**-space integrals using the Bargmann measure.

3. Matrix elements of the boson operators

3.1. The boson operators in generalized VCS theory

In generalized VCS theory (Hecht 1989, Hecht and Biedenharn 1990, Hecht and Chen 1990), equation (2.20) is applied not only to the Lie algebra generators, but also to operators lying outside the algebra. In the realization (2.2) of sp(2N, ℝ), the simplest operators of this kind are the boson creation and annihilation operators $a^\dagger_{is}$, a_{is}. From their commutators with the sp(2N, ℝ) generators

$$\begin{aligned} &[E_{ij}, a^\dagger_{ks}] = \delta_{jk}a^\dagger_{is} \qquad [E_{ij}, a_{ks}] = -\delta_{ik}a_{js} \\ &[D^\dagger_{ij}, a^\dagger_{ks}] = 0 \qquad [D^\dagger_{ij}, a_{ks}] = -\delta_{ik}a^\dagger_{js} - \delta_{jk}a^\dagger_{is} \\ &[D_{ij}, a^\dagger_{ks}] = \delta_{ik}a_{js} + \delta_{jk}a_{is} \qquad [D_{ij}, a_{ks}] = 0 \end{aligned} \tag{3.1}$$

it follows that for any given value of index s, they are vector operators or, in other words, they are the components of an $\mathrm{sp}(2N,\mathbb{R})$ irreducible tensor $T^{(1)}$ transforming under the $2N$-dimensional (non-unitary) irrep $\langle 1\rangle$. In an $\mathrm{sp}(2N,\mathbb{R})\supset \mathrm{u}(N)$ classification scheme, one has

$$T^{(1)}_{\{1\}i}=a_i^\dagger \qquad T^{(1)}_{\{-1\}-i}=\tilde{a}_{-i}=(-1)^{i-1}a_i \tag{3.2}$$

where, for simplicity's sake, we have dropped index s, which is assumed to be fixed, and where label i (resp $-i$) corresponds to a $\mathrm{u}(N)$ weight $\Delta^{(1)}(i)=(0\ldots010\ldots0)$ (resp. $-\Delta^{(1)}(i)=(0\ldots0-10\ldots0)$ with 1 (resp. -1) appearing in position i.

Under Hermitian conjugation, the $\mathrm{sp}(2N,\mathbb{R})$ irreducible tensor $T^{(1)}$ satisfies the symmetry property

$$(T^{(1)}_{\{q\}\kappa})^\dagger=(-1)^{\varepsilon(\langle 1\rangle,\{q\})+\delta(\{q\},\kappa)}T^{(1)}_{\{\bar{q}\}\bar{\kappa}} \tag{3.3}$$

where $\{\bar{q}\}\bar{\kappa}=\{-1\}-i$ or $\{1\}i$ according to whether $\{q\}\kappa=\{1\}i$ or $\{-1\}-i$, and $\varepsilon(\langle 1\rangle,\{q\})$, $\delta(\{q\},\kappa)$ are $\mathrm{sp}(2N,\mathbb{R})$: $\mathrm{u}(N)$ and $\mathrm{u}(N)$ Hermitian conjugation factors respectively. The latter, generalizing the su(2) conjugation factor $\delta(j,m)=j-m$, is given in the present case by (Baird and Biedenharn 1964)

$$\delta(\{1\},i)=i-1 \qquad \delta(\{-1\},-i)=N-i \tag{3.4}$$

while the former is defined by

$$\varepsilon(\langle 1\rangle,\{1\})=0 \qquad \varepsilon(\langle 1\rangle,\{-1\})=-N+1 \tag{3.5}$$

so as to reproduce (3.2). Note that $\delta(\{q\},\kappa)$ fulfils the condition

$$\delta(\{1\},i)+\delta(\{-1\},-i)=2\varphi(\{1\})=N-1 \tag{3.6}$$

where the functional

$$\varphi(\{q\})=\tfrac{1}{2}\sum_i (N+1-2i)q_i \tag{3.7}$$

reducing to j for su(2) is used to define $\mathrm{u}(N)$ phase factors (Hecht *et al* 1987).

From (2.7) and (3.1), the VCS representation Γ of the boson operators is given by

$$\Gamma(a_i^\dagger)=\mathbb{a}_i^\dagger+\sqrt{N+1}\,[\tilde{\mathbb{a}}\times P^{(2)}(\mathbf{z})]^{\{1\}}_i \qquad \Gamma(\tilde{a}_{-i})=\tilde{\mathbb{a}}_{-i}. \tag{3.8a, b}$$

It is expressed in terms of two types of operators, the $\mathbf{z}$-space polynomial tensor $P^{(2)}(\mathbf{z})$, and the intrinsic boson operators $\mathbb{a}_i^\dagger$, $\tilde{\mathbb{a}}_{-i}$. The components of the former

$$P^{(2)}_{ij}(\mathbf{z})=(1+\delta_{ij})^{-1/2}z_{ij} \tag{3.9}$$

are labelled by ij, denoting the $\mathrm{u}(N)$ weight $\Delta^{(2)}(i,j)=(0\ldots020\ldots0)$ with 2 in position i whenever $i=j$, or $(0\ldots010\ldots010\ldots0)$ with 1 in positions i and j whenever $i\neq j$. The latter are assumed to commute with $\mathbf{z}$ and $\boldsymbol{\nabla}$ and are defined through their left action on intrinsic states. This means that they must always be commuted through to the left so that they can act on the adjacent intrinsic bra. In (3.8), the $\mathrm{u}(N)$ Wigner coefficients for the coupling of $\{2\}$ and $\{-1\}$ are obtained from Biedenharn and Louck (1968). As it happens for the $\mathrm{sp}(2N,\mathbb{R})$ generators, $\Gamma(a_i^\dagger)$ and $\Gamma(\tilde{a}_{-i})$ form a non-unitary realization of the boson operators. A unitary realization $\gamma(a_i^\dagger)$, $\gamma(\tilde{a}_{-i})$ can be obtained through (2.12).

Unlike the intrinsic $\mathrm{u}(N)$ generators $\mathbb{E}_{ij}$, which connect intrinsic bras $\langle\{\omega\}\alpha'|$ only to intrinsic kets $|\{\omega\}\alpha\rangle$, the intrinsic boson operators $\mathbb{a}_i^\dagger$, $\tilde{\mathbb{a}}_{-i}$ can convert $\langle\{\omega'\}\alpha'|$ into an intrinsic or non-intrinsic ket belonging to an irrep $\langle\omega\rangle\neq\langle\omega'\rangle$. Hence the first step in the calculation of the reduced matrix elements of the boson operators consists in evaluating those of the intrinsic operators between the purely intrinsic bras and the permitted kets. Our first objective will therefore be to determine for which irreps $\{\omega'\}$, $\{\nu\}$, $\{h\}$, the reduced matrix elements $(\{\omega'\}\|\tilde{\mathfrak{a}}\|\langle\omega\rangle\{\nu\}\rho\{h\})$ and $(\{\omega'\}\|\mathfrak{a}^{\dagger}\|\langle\omega\rangle\{\nu\}\rho\{h\})$ are different from zero for a given $\langle\omega\rangle$, and to calculate the non-vanishing ones in terms of $\langle\{\omega'\}\|\tilde{a}\|\{\omega\}\rangle$ and $\langle\{\omega'\}\|a^{\dagger}\|\{\omega\}\rangle$. Note that we are here in a case where equation (2.20) can be applied as it stands because the coupling of any u(N) irrep with the irreps {1} and {−1} characterizing the boson operator transformation properties is multiplicity free.

3.2. Reduced matrix elements of the intrinsic boson operators

From (2.20), (3.8b), and the K-matrix normalization condition $K(\{\omega\}, \{\omega\})=1$, we obtain the relation

$$\langle\{\omega'\}\|\tilde{a}\|\langle\omega\rangle\{h\}r\rangle=\sum_{\{\nu\}\rho}(\{\omega'\}\|\tilde{\mathfrak{a}}\|\langle\omega\rangle\{\nu\}\rho\{h\})(K(\{\omega\},\{h\}))_{\{\nu\}\rho,r}. \tag{3.10}$$

On the other hand, from (2.19) the left-hand side of (3.10) can be written as

$$\langle\{\omega'\}\|\tilde{a}\|\langle\omega\rangle\{h\}r\rangle=\sum_{\{\nu\}\rho}\langle\{\omega'\}\|\tilde{a}\|\langle\omega\rangle\{\nu\}\rho\{h\}\rangle((K^{\dagger}(\{\omega\},\{h\}))^{-1})_{\{\nu\}\rho,r}. \tag{3.11}$$

For $\{\nu\}=\{0\}$, the reduced matrix element on the right-hand side of (3.11) is simply $\langle\{\omega'\}\|\tilde{a}\|\{\omega\}\rangle$, where $\{\omega'\}$ must be of the form $\{\omega'\}=\{\omega-\Delta^{(1)}(i)\}$ with $i\in\{1,\ldots,N\}$. For $\{\nu\}\neq\{0\}$, applying (2.3) enables the reduced matrix element to be written as

$$\begin{aligned}\langle\{\omega'\}\|\tilde{a}\|\langle\omega\rangle\{\nu\}\rho\{h\}\rangle&\\ &=\langle\{\omega'\}\alpha'|[\tilde{a}\times[P^{\{\nu\}}(\mathbf{D}^{\dagger})\times|\{\omega\}\rangle]^{\rho\{h\}}]^{\{\omega'\}}_{\alpha'}\\ &=\sum_{\{\mu\}\rho'}U(\{\omega\}\{\nu\}\{\omega'\}\{-1\};\{h\}\rho\{\mu\}\rho')\\ &\quad\times\langle\{\omega'\}\alpha'|[[\tilde{a},P^{\{\nu\}}(\mathbf{D}^{\dagger})]^{\{\mu\}}\times|\{\omega\}\rangle]^{\rho'\{\omega'\}}_{\alpha'}\end{aligned} \tag{3.12}$$

where the U coefficient is a u(N) Racah coefficient in unitary form (Hecht *et al* 1981, 1987). In the last step, use has been made of the fact that $D^{\dagger}_{ij}$ annihilates the intrinsic bra $\langle\{\omega'\}\alpha'|$ to replace the coupled product $[\tilde{a}\times P^{\{\nu\}}(\mathbf{D}^{\dagger})]^{\{\mu\}}$ by a coupled commutator $[\tilde{a}, P^{\{\nu\}}(\mathbf{D}^{\dagger})]^{\{\mu\}}$. Since $P^{\{\nu\}}(\mathbf{D}^{\dagger})$ is a polynomial of degree $\frac{1}{2}\sum_i\nu_i$ in $D^{\dagger}_{ij}$, it results from (3.1) that such a coupled commutator is equal to a boson creation operator multiplied by some polynomial of degree $\frac{1}{2}\sum_i\nu_i-1$ in $D^{\dagger}_{ij}$. Hence the reduced matrix element (3.12) vanishes if $\{\nu\}\neq\{2\}$ and, in the opposite case, reduces to

$$\langle\{\omega'\}\|\tilde{a}\|\langle\omega\rangle\{2\}\{h\}\rangle=\sqrt{N+1}\,U(\{\omega\}\{2\}\{\omega'\}\{-1\};\{h\}\{1\})\langle\{\omega'\}\|a^{\dagger}\|\{\omega\}\rangle \tag{3.13}$$

because

$$[\tilde{a}, P^{\{2\}}(\mathbf{D}^{\dagger})]^{\{1\}}_i=\sqrt{N+1}\,a^{\dagger}_i. \tag{3.14}$$

In (3.13), the irreps $\{\omega'\}$ and $\{h\}$ must be of the form $\{\omega'\}=\{\omega+\Delta^{(1)}(i)\}$ and $\{h\}=\{\omega+\Delta^{(2)}(i,j)\}$, where $i,j\in\{1,\ldots,N\}$.

By combining these results with (3.10) and (3.11), we conclude that the only non-vanishing reduced matrix elements of the intrinsic boson annihilation operators are given by

$$(\{\omega-\Delta^{(1)}(i)\}\|\tilde{\mathfrak{a}}\|\{\omega\})=\langle\{\omega-\Delta^{(1)}(i)\}\|\tilde{a}\|\{\omega\}\rangle \tag{3.15}$$

and

$$(\{\omega+\Delta^{(1)}(i)\}\|\tilde{\mathfrak{a}}\|\langle\omega\rangle\{\omega+\Delta^{(2)}(i,j)\})$$
$$=\sqrt{N+1}\,K^{-2}(\{\omega\},\{\omega+\Delta^{(2)}(i,j)\})$$
$$\times U(\{\omega\}\{2\}\{\omega+\Delta^{(1)}(i)\}\{-1\};\{\omega+\Delta^{(2)}(i,j)\}\{1\})$$
$$\times\langle\{\omega+\Delta^{(1)}(i)\}\|a^{\dagger}\|\{\omega\}\rangle \tag{3.16}$$

where, on the left-hand side of (3.16), we have dropped the label $r=1$, which is uniquely determined. Finally, by using (A5), the symmetry properties of multiplicity-free Racah coefficients (Hecht *et al* 1981), and some results of Le Blanc and Hecht (1987), we obtain the following explicit expression for the reduced matrix element (3.16):

$$(\{\omega+\Delta^{(1)}(i)\}\|\tilde{\mathfrak{a}}\|\langle\omega\rangle\{\omega+\Delta^{(2)}(i,j)\})$$
$$=\frac{(-1)^{j-1}}{\omega_i+\omega_j-i-j+1+\delta_{ij}}\left[(1+\delta_{ij})\prod_{k\neq j}\frac{\omega_k-\omega_j+j-k-1-\delta_{ij}}{\omega_k-\omega_j+j-k+\delta_{ki}-\delta_{ij}}\right]^{1/2}$$
$$\times\langle\{\omega+\Delta^{(1)}(i)\}\|a^{\dagger}\|\{\omega\}\rangle. \tag{3.17}$$

By proceeding in the same way for the boson creation operators, we obtain for the counterparts of (3.10) and (3.11)

$$\langle\{\omega'\}\|a^{\dagger}\|\langle\omega\rangle\{h\}r\rangle$$
$$=\sum_{\{\nu\}\rho}[(\{\omega'\}\|\mathfrak{a}^{\dagger}\|\langle\omega\rangle\{\nu\}\rho\{h\})$$
$$+\sqrt{N+1}\,(\{\omega'\}\|[\tilde{\mathfrak{a}}\times P^{(2)}(\mathbf{z})]^{\{1\}}\|\langle\omega\rangle\{\nu\}\rho\{h\})](K(\{\omega\},\{h\}))_{\{\nu\}\rho,r} \tag{3.18}$$

and

$$\langle\{\omega'\}\|a^{\dagger}\|\langle\omega\rangle\{h\}r\rangle=\sum_{\{\nu\}\rho}\langle\{\omega'\}\|a^{\dagger}\|\langle\omega\rangle\{\nu\}\rho\{h\}\rangle((K^{\dagger}(\{\omega\},\{h\}))^{-1})_{\{\nu\}\rho,r} \tag{3.19}$$

respectively.

The second term on the right-hand side of (3.18) can be transformed.

$$(\{\omega'\}\|[\tilde{\mathfrak{a}}\times P^{(2)}(\mathbf{z})]^{\{1\}}\|\langle\omega\rangle\{\nu\}\rho\{h\})$$
$$=\sum_{\{h'\}}U(\{h\}\{2\}\{\omega'\}\{-1\};\{h'\}\{1\})$$
$$\times(\{\omega'\}\alpha'|[\tilde{\mathfrak{a}}\times[P^{(2)}(\mathbf{z})\times|\langle\omega\rangle\{\nu\}\rho\{h\})]^{\{h'\}}]^{\{\omega'\}}_{\alpha'} \tag{3.20}$$

where

$$[P^{(2)}(\mathbf{z})\times|\langle\omega\rangle\{\nu\}\rho\{h\})]^{\{h'\}}_{\chi'}$$
$$=\sum_{\{\nu'\}\rho'}|\langle\omega\rangle\{\nu'\}\rho'\{h'\}\chi')(\langle\omega\rangle\{\nu'\}\rho'\{h'\}\|z\|\langle\omega\rangle\{\nu\}\rho\{h\}). \tag{3.21}$$

In (3.21), the reduced matrix element of $P^{(2)}(\mathbf{z})=\mathbf{z}$ is given by (Le Blanc and Rowe 1987)

$$(\langle\omega\rangle\{\nu'\}\rho'\{h'\}\|z\|\langle\omega\rangle\{\nu\}\rho\{h\})=U(\{\omega\}\{\nu\}\{h'\}\{2\};\{h\}\rho\{\nu'\}\rho')(\{\nu'\}\|z\|\{\nu\}) \tag{3.22}$$

where $(\{\nu'\}\|z\|\{\nu\})$ has the value

$$(\{\nu'\}\|z\|\{\nu\})=\left[\tfrac{1}{2}(\nu_k+N+2-k)\prod_{j\neq k}\frac{\nu_k-\nu_j+j-k+1}{\nu_k-\nu_j+j-k+2}\right]^{1/2} \tag{3.23}$$

whenever $\{\nu'\}=\{\nu+2\Delta^{(1)}(k)\}$ for some $k\in\{1,\ldots,N\}$, and zero otherwise. By combining these results with (3.15) and (3.16), it is clear that the only non-vanishing reduced matrix element (3.20) corresponds to $\{\nu\}=\{0\}$ (thence $\{\nu'\}=\{2\}$ in (3.21)), $\{h\}=\{\omega\}$, $\{\omega'\}=\{\omega+\Delta^{(1)}(i)\}$, and that the summation over $\{h'\}$ runs over $\{h'\}=\{\omega+\Delta^{(2)}(i,j)\}$. Hence we obtain

$$\begin{aligned}(\{\omega+\Delta^{(1)}(i)\}\|[\bar{\mathfrak{a}}\times P^{(2)}(\mathbf{z})]^{(1)}\|\{\omega\}) &= \sqrt{N+1}\Big[\sum_j K^{-2}(\{\omega\},\{\omega+\Delta^{(2)}(i,j)\}) \\ &\quad\times U^2(\{\omega\}\{2\}\{\omega+\Delta^{(1)}(i)\}\{-1\};\{\omega+\Delta^{(2)}(i,j)\}\{1\})\Big] \\ &\quad\times\langle\{\omega+\Delta^{(1)}(i)\}\|a^\dagger\|\{\omega\}\rangle. \end{aligned}\tag{3.24}$$

On the other hand, the reduced matrix element on the right-hand side of (3.19) can be written as

$$\langle\{\omega'\}\|a^\dagger\|\langle\omega\rangle\{\nu\}\rho\{h\}\rangle=\langle\{\omega'\}\alpha'|[\mathfrak{a}^\dagger\times[P^{\{\nu\}}(\mathbf{D}^\dagger)\times|\{\omega\}\rangle]^{\rho\{h\}}]^{\{\omega'\}}_{\alpha'}.\tag{3.25}$$

Since $P^{\{\nu\}}(\mathbf{D}^\dagger)$ commutes with $\mathfrak{a}^\dagger$ and for $\{\nu\}\neq\{0\}$ annihilates the intrinsic bra $\langle\{\omega'\}\alpha'|$, the only non-vanishing result is obtained for $\{\nu\}=\{0\}$, $\{h\}=\{\omega\}$, and $\{\omega'\}=\{\omega+\Delta^{(1)}(i)\}$, and is just $\langle\{\omega'\}\|a^\dagger\|\{\omega\}\rangle$.

By combining (3.18) and (3.19) with (3.24) and (3.25), we conclude that the only non-vanishing reduced matrix elements of the intrinsic boson creation operators are given by

$$\begin{aligned}(\{\omega+\Delta^{(1)}(i)\}\|\mathfrak{a}^\dagger\|\{\omega\}) &= \Big[1-(N+1)\sum_j K^{-2}(\{\omega\},\{\omega+\Delta^{(2)}(i,j)\}) \\ &\quad\times U^2(\{\omega\}\{2\}\{\omega+\Delta^{(1)}(i)\}\{-1\};\{\omega+\Delta^{(2)}(i,j)\}\{1\})\Big]\langle\{\omega+\Delta^{(1)}(i)\}\|a^\dagger\|\{\omega\}\rangle \\ &= \left[1-\sum_j\frac{1+\delta_{ij}}{\omega_i+\omega_j-i-j+1+\delta_{ij}}\prod_{k\neq j}\frac{\omega_k-\omega_j+j-k-1-\delta_{ij}}{\omega_k-\omega_j+j-k+\delta_{ki}-\delta_{ij}}\right] \\ &\quad\times\langle\{\omega+\Delta^{(1)}(i)\}\|a^\dagger\|\{\omega\}\rangle.\end{aligned}\tag{3.26}$$

The sum over j in (3.26) can be easily performed by using complex function residue theory (Le Blanc and Rowe 1987, Quesne 1990a), so that we finally obtain

$$(\{\omega+\Delta^{(1)}(i)\}\|\mathfrak{a}^\dagger\|\{\omega\})=\left(\prod_k\frac{\omega_i+\omega_k-i-k}{\omega_i+\omega_k-i-k+1+\delta_{ki}}\right)\langle\{\omega+\Delta^{(1)}(i)\}\|a^\dagger\|\{\omega\}\rangle.\tag{3.27}$$

From (3.15), (3.17) and (3.27), we see that the only intrinsic bras that the intrinsic boson operators can connect to some ket belonging to a given sp($2N,\mathbb{R}$) irrep $\langle\omega\rangle$ are $\langle\{\omega-\Delta^{(1)}(i)\}\alpha'|$ and $\langle\{\omega+\Delta^{(1)}(i)\}_{\alpha'}|$ for $i=1,2,\ldots,N$. Since the $\mathbf{z}$-space polynomial tensor appearing in the VCS representation (3.8) of the boson operators cannot change the sp($2N,\mathbb{R}$) irrep and the transition from such an irrep to another one is entirely due to the intrinsic boson operators, we conclude that the only non-vanishing reduced matrix elements of the boson operators are $\langle\langle\omega'\rangle\{h'\}r'\|a\|\langle\omega\rangle\{h\}r\rangle$ and $\langle\langle\omega'\rangle\{h'\}r'\|a^\dagger\|\langle\omega\rangle\{h\}r\rangle$, where $\langle\omega'\rangle=\langle\omega-\Delta^{(1)}(i)\rangle$, $\langle\omega+\Delta^{(1)}(i)\rangle$, and $\{h'\}=\{h-\Delta^{(1)}(j)\}$ for the former or $\{h'\}=\{h+\Delta^{(1)}(j)\}$ for the latter. In the next subsection, we shall proceed to calculate them in terms of pure intrinsic space reduced matrix elements.

3.3. Reduced matrix elements of the boson operators

Let us first consider the boson operator reduced matrix elements corresponding to $\langle\omega'\rangle = \langle\omega - \Delta^{(1)}(i)\rangle$. From (2.20) and (3.8$b$), those of the annihilation operators are given by

$$\begin{aligned}\langle\langle\omega - \Delta^{(1)}(i)\rangle\{h - \Delta^{(1)}(j)\}r'\|\tilde{a}\|\langle\omega\rangle\{h\}r\rangle &= \sum_{\{\iota\}\rho}\sum_{\{\nu\}\rho} (K^{-1}(\{\omega - \Delta^{(1)}(i)\}, \{h - \Delta^{(1)}(j)\}))_{r'\{\nu'\}\rho'} \\ &\quad\times (\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h - \Delta^{(1)}(j)\}\|\Gamma(\tilde{a})\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ &\quad\times (K(\{\omega\}, \{h\}))_{\{\nu\}\rho r}\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}(\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h - \Delta^{(1)}(j)\}\|\Gamma(\tilde{a})\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ = (\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h - \Delta^{(1)}(j)\}\|\tilde{\mathbf{a}}\|\langle\omega\rangle\{\nu\}\rho\{h\}).\end{aligned} \tag{3.29}$$

By using standard $\mathrm{u}(N)$ recoupling techniques and by taking (3.15) into account, equation (3.29) becomes

$$\begin{aligned}(\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h - \Delta^{(1)}(j)\}\|\Gamma(\tilde{a})\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ = \delta_{\{\nu'\},\{\nu\}}(-1)^{\varphi(\{\omega\})+\varphi(\{h-\Delta^{(1)}(j)\})-\varphi(\{\omega-\Delta^{(1)}(i)\})-\varphi(\{h\})} \\ \times U(\{\nu\}\{\omega\}\{h - \Delta^{(1)}(j)\}\{-1\}; \{h\}\rho\{\omega - \Delta^{(1)}(i)\}\rho') \\ \times \langle\{\omega - \Delta^{(1)}(i)\}\|\tilde{a}\|\{\omega\}\rangle\end{aligned} \tag{3.30}$$

where the $\mathrm{u}(N)$ phase factor φ has been defined in (3.7).

By proceeding in the same way for the creation operators, we obtain

$$\begin{aligned}\langle\langle\omega - \Delta^{(1)}(i)\rangle\{h + \Delta^{(1)}(j)\}r'\|a^{\dagger}\|\langle\omega\rangle\{h\}r\rangle &= \sum_{\{\iota\}\rho}\sum_{\{\iota\}\rho} (K^{-1}(\{\omega - \Delta^{(1)}(i)\}, \{h + \Delta^{(1)}(j)\}))_{r',\{\nu'\}\rho'} \\ &\quad\times (\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h + \Delta^{(1)}(j)\}\|\Gamma(a^{\dagger})\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ &\quad\times (K(\{\omega\}, \{h\}))_{\{\iota\}\rho,r}\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}(\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h + \Delta^{(1)}(j)\}\|\Gamma(a^{\dagger})\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ = \sqrt{N+1}\,(\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h + \Delta^{(1)}(j)\} \\ \times \|[\tilde{\mathbf{a}} \times P^{\{2\}}(\mathbf{z})]^{\{1\}}\|\langle\omega\rangle\{\nu\}\rho\{h\}).\end{aligned} \tag{3.32}$$

From the results of the previous subsection, it indeed follows that the intrinsic boson creation operator on the right-hand side of (3.8a) does not contribute to the reduced matrix element (3.32). By changing the order of $\mathbf{a}$ and $P^{\{2\}}(\mathbf{z})$ in the latter, thereby introducing a phase factor $(-1)^{N-1}$, and by using standard recoupling techniques, we get

$$\begin{aligned}(\langle\omega - \Delta^{(1)}(i)\rangle\{\nu'\}\rho'\{h + \Delta^{(1)}(j)\}\|\Gamma(a^{\dagger})\|\langle\omega\rangle\{\nu\}\rho\{h\}) \\ = (-1)^{N-1}\sqrt{N+1}\begin{bmatrix} \{\omega\} & \{\nu\} & \{h\}\rho \\ \{-1\} & \{2\} & \{1\} \\ \{\omega - \Delta^{(1)}(i)\} & \{\nu'\} & \{h + \Delta^{(1)}(j)\}\rho' \end{bmatrix}(\{\nu'\}\|z\|\{\nu\}) \\ \times \langle\{\omega - \Delta^{(1)}(i)\}\|\tilde{a}\|\{\omega\}\rangle\end{aligned} \tag{3.33}$$

where the [ ] symbol is a u(N) 9j-type symbol in unitary form. The reduced matrix element (3 33) has therefore been factorized into a **z**-space reduced matrix element of **z**, whose explicit form is given in (3.23), and a pure intrinsic space boson operator reduced matrix element.

The boson operator reduced matrix elements corresponding to $\{\omega'\}=\{\omega+\Delta^{(1)}(i)\}$ could be similarly calculated. Simpler expressions are, however, obtained through the Hermitian conjugation of the reduced matrix elements corresponding to $\{\omega'\}=\{\omega-\Delta^{(1)}(i)\}$,

$$\langle\langle\omega+\Delta^{(1)}(i)\rangle\{h'\}r'\|T^{(1)}_{\{q\}}\|\langle\omega\rangle\{h\}r\rangle$$
$$=(-1)^{\varphi(\{h\})+\varphi(\{q\})-\varphi(\{h'\})+\varepsilon(\langle 1\rangle,\{q\})}\left[\frac{\dim\{h\}}{\dim\{h'\}}\right]^{1/2}$$
$$\times\langle\langle\omega\rangle\{h\}r\|T^{(1)}_{\{\bar q\}}\|\langle\omega+\Delta^{(1)}(i)\rangle\{h'\}r'\rangle. \tag{3.34}$$

Here $\dim\{h\}$ denotes the dimension of the u(N) irrep $\{h\}$, given by

$$\dim\{h\}=[1!2!\ldots(N-1)!]^{-1}\prod_{r<s}(h_r-h_s+s-r) \tag{3.35}$$

while the phase factors $(-1)^{\varphi(\{h\})+\varphi(\{q\})-\varphi(\{h'\})}$ and $(-1)^{\varepsilon(\langle 1\rangle,\{q\})}$ follow from the $1\leftrightarrow 3$ interchange symmetry property of the u(N) Wigner coefficients and from the Hermitian conjugation symmetry property (3 3), respectively. By collecting the results contained in (3.5), (3.7) and (3 35), equation (3.24) can be rewritten as

$$\langle\langle\omega+\Delta^{(1)}(i)\rangle\{h-\Delta^{(1)}(j)\}r'\|\tilde a\|\langle\omega\rangle\{h\}r\rangle$$
$$=(-1)^{i-j}\left[\prod_{k\neq j}\frac{h_k-h_j+j-k}{h_k-h_j+j-k+1}\right]^{1/2}$$
$$\times\langle\langle\omega\rangle\{h\}r\|a^\dagger\|\langle\omega+\Delta^{(1)}(i)\rangle\{h-\Delta^{(1)}(j)\}r'\rangle \tag{3.36}$$

and

$$\langle\langle\omega+\Delta^{(1)}(i)\rangle\{h+\Delta^{(1)}(j)\}r'\|a^\dagger\|\langle\omega\rangle\{h\}r\rangle$$
$$=(-1)^{j-i}\left[\prod_{k\neq j}\frac{h_k-h_j+j-k}{h_k-h_j+j-k-1}\right]^{1/2}$$
$$\times\langle\langle\omega\rangle\{h\}r\|\tilde a\|\langle\omega+\Delta^{(1)}(i)\rangle\{h+\Delta^{(1)}(j)\}r'\rangle. \tag{3.37}$$

4. sp$(2N,\mathbb{R})\supset$ u(N) reduced Wigner coefficients

The Wigner–Eckart theorem with respect to sp$(2N,\mathbb{R})\supset$ u(N) can now be used to express the u(N) reduced matrix elements of $T^{(1)}_{\{q\}}$ in terms of its sp$(2N,\mathbb{R})$ triple reduced matrix elements and sp$(2N,\mathbb{R})\supset$ u(N) reduced Wigner coefficients coupling a positive discrete series irrep $\langle\omega\rangle$ with the vector irrep $\langle 1\rangle$:

$$\langle\langle\omega'\rangle\{h'\}r'\|T^{(1)}_{\{q\}}\|\langle\omega\rangle\{h\}r\rangle=\langle\langle\omega'\rangle\|\!|T^{(1)}|\!\|\langle\omega\rangle\rangle\langle\langle\omega\rangle\{h\}r;\langle 1\rangle\{q\}\|\langle\omega'\rangle\{h'\}r'\rangle. \tag{4.1}$$

Since the pure intrinsic space reduced matrix element $\langle\{\omega-\Delta^{(1)}(i)\}\|\tilde{a}\|\{\omega\}\rangle$, considered in the previous section, can be similarly written in terms of a triple reduced matrix element of $T^{(1)}$ and a special reduced Wigner coefficient $\langle\langle\omega\rangle\{\omega\};\langle1\rangle\{-1\}\|\langle\omega-\Delta^{(1)}(i)\rangle\{\omega-\Delta^{(1)}(i)\}\rangle$, the former can be eliminated from the formulae and the general reduced Wigner coefficients expressed in terms of the special one, playing the role of a normalization constant. Apart from this unknown constant, the $\mathrm{sp}(2N,\mathbb{R})\supset \mathrm{u}(N)$ reduced Wigner coefficients are therefore entirely determined by the results of the previous section.

Since the $\mathrm{sp}(2N,\mathbb{R})$ irrep $\langle1\rangle$ is not unitary, the corresponding reduced Wigner coefficients do not fulfil any orthogonality relation, which would make it possible to normalize them. Instead of choosing a somewhat arbitrary normalization convention, as was done by Ui (1968) for $\mathrm{sp}(2,\mathbb{R})$, we shall leave it undetermined and only consider ratios of reduced Wigner coefficients

$$R_{r'r}=\frac{\langle\langle\omega\rangle\{h\}r,\langle1\rangle\{q\}\|\langle\omega'\rangle\{h'\}r'\rangle}{\langle\langle\omega\rangle\{\omega\};\langle1\rangle\{\bar{q}\}\|\langle\omega'\rangle\{\omega'\}\rangle}. \tag{4.2}$$

This is no practical limitation because only such ratios do appear in applications Note that in (4.2), $\{h'\}=\{h-\Delta^{(1)}(j)\}$ or $\{h+\Delta^{(1)}(j)\}$ according to whether $\{q\}=\{-1\}$ or $\{1\}$, and $\{\bar{q}\}=\{-1\}$ or $\{1\}$ according to whether $\langle\omega'\rangle=\langle\omega-\Delta^{(1)}(i)\rangle$ or $\langle\omega+\Delta^{(1)}(i)\rangle$.

For $\langle\omega'\rangle=\langle\omega-\Delta^{(1)}(i)\rangle$, the results are

$$\begin{aligned}&\frac{\langle\langle\omega\rangle\{h\}r;\langle1\rangle\{-1\}\|\langle\omega-\Delta^{(1)}(i)\rangle\{h-\Delta^{(1)}(j)\}r'\rangle}{\langle\langle\omega\rangle\{\omega\};\langle1\rangle\{-1\}\|\langle\omega-\Delta^{(1)}(i)\rangle\{\omega-\Delta^{(1)}(i)\}\rangle}\\&\quad=(-1)^{j-i}\sum_{\{\nu\}\rho\rho'}(K^{-1}(\{\omega-\Delta^{(1)}(i)\},\{h-\Delta^{(1)}(j)\}))_{r',\{\nu\}\rho}\\&\qquad\times U(\{\nu\}\{\omega\}\{h-\Delta^{(1)}(j)\}\{-1\};\{h\}\rho\{\omega-\Delta^{(1)}(i)\}\rho')(K(\{\omega\},\{h\}))_{\{\nu\}\rho,r}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\frac{\langle\langle\omega\rangle\{h\}r;\langle1\rangle\{1\}\|\langle\omega-\Delta^{(1)}(i)\rangle\{h+\Delta^{(1)}(j)\}r'\rangle}{\langle\langle\omega\rangle\{\omega\},\langle1\rangle\{-1\}\|\langle\omega-\Delta^{(1)}(i)\rangle\{\omega-\Delta^{(1)}(i)\}\rangle}\\&\quad=(-1)^{N-1}\sqrt{N+1}\sum_{\{\nu\}\{\nu'\}\rho\rho}(K^{-1}(\{\omega-\Delta^{(1)}(i)\},\{h+\Delta^{(1)}(j)\}))_{r',\{\nu'\}\rho'}\\&\qquad\times\begin{bmatrix}\{\omega\} & \{\nu\} & \{h\}\rho\\ \{-1\} & \{2\} & \{1\}\\ \{\omega-\Delta^{(1)}(i)\} & \{\nu'\} & \{h+\Delta^{(1)}(j)\}\rho'\end{bmatrix}\\&\qquad\times(\{\nu'\}\|z\|\{\nu\})(K(\{\omega\},\{h\}))_{\{\nu\}\rho,r}.\end{aligned} \tag{4.4}$$

The remaining ratios of reduced Wigner coefficients, corresponding to $\langle\omega'\rangle=\langle\omega+\Delta^{(1)}(i)\rangle$, can be calculated through the symmetry relations

$$\begin{aligned}&\frac{\langle\langle\omega\rangle\{h\}r;\langle1\rangle\{-1\}\|\langle\omega+\Delta^{(1)}(i)\rangle\{h-\Delta^{(1)}(j)\}r'\rangle}{\langle\langle\omega\rangle\{\omega\};\langle1\rangle\{1\}\|\langle\omega+\Delta^{(1)}(i)\rangle\{\omega+\Delta^{(1)}(i)\}\rangle}\\&\quad=(-1)^{i+j}\left[\left(\prod_{k\neq i}\frac{\omega_k-\omega_i+i-k-1}{\omega_k-\omega_i+i-k}\right)\left(\prod_{k\neq j}\frac{h_k-h_j+j-k}{h_k-h_j+j-k+1}\right)\right]^{1/2}\\&\qquad\times\frac{\langle\langle\omega+\Delta^{(1)}(i)\rangle\{h-\Delta^{(1)}(j)\}r';\langle1\rangle\{1\}\|\langle\omega\rangle\{h\}r\rangle}{\langle\langle\omega+\Delta^{(1)}(i)\rangle\{\omega+\Delta^{(1)}(i)\};\langle1\rangle\{-1\}\|\langle\omega\rangle\{\omega\}\rangle}\end{aligned} \tag{4.5}$$

and

$$\frac{\langle\langle\omega\rangle\{h\}r;\langle 1\rangle\{1\}\|\langle\omega+\Delta^{(1)}(i)\rangle\{h+\Delta^{(1)}(j)\}r'\rangle}{\langle\langle\omega\rangle\{\omega\};\langle 1\rangle\{1\}\|\langle\omega+\Delta^{(1)}(i)\rangle\{\omega+\Delta^{(1)}(i)\}\rangle}$$
$$=(-1)^{i+j}\left[\left(\prod_{k\neq i}\frac{\omega_k-\omega_i+i-k-1}{\omega_k-\omega_i+i-k}\right)\left(\prod_{k\neq j}\frac{h_k-h_j+j-k}{h_k-h_j+j-k-1}\right)\right]^{1/2}$$
$$\times\frac{\langle\langle\omega+\Delta^{(1)}(i)\rangle\{h+\Delta^{(1)}(j)\}r',\langle 1\rangle\{-1\}\|\langle\omega\rangle\{h\}r\rangle}{\langle\langle\omega+\Delta^{(1)}(i)\rangle\{\omega+\Delta^{(1)}(i)\};\langle 1\rangle\{-1\}\|\langle\omega\rangle\{\omega\}\rangle} \tag{4.6}$$

resulting from (3 36) and (3 37)

Table 1. Ratios R of $\mathrm{sp}(2N,\mathbb{R})\supset \mathrm{u}(N)$ reduced Wigner coefficients for the coupling $\langle\omega\rangle\times\langle 1\rangle\to\langle\omega'\rangle$ Here $h_k=\omega+\nu_k$, $k=1,\ \ldots, N$, where ν_k is some non-negative even integer

$\{q\}$	$\langle\omega'\rangle$	R
$\{-1\}$	$\langle\omega\,\omega-1\rangle$	$(-1)^{N-j}\left[\frac{h_j+\omega-j-1}{2\omega-N-1}\right]^{1/2}$
$\{-1\}$	$\langle\omega+1\,\omega\rangle$	$\left[\frac{h_j-\omega+N-j}{2\omega}\right]^{1/2}$
$\{1\}$	$\langle\omega\,\omega-1\rangle$	$(-1)^{N-i}\left[\frac{h_i-\omega+N-j+2}{2\omega-N-1}\right]^{1/2}$
$\{1\}$	$\langle\omega+1\,\omega\rangle$	$(-1)^{j-1}\left[\frac{h_i+\omega-j+1}{2\omega}\right]^{1/2}$

Table 2 Ratios R of $\mathrm{sp}(2N,\mathbb{R})\supset \mathrm{u}(N)$ reduced Wigner coefficients for the coupling $\langle\omega\rangle\times\langle 1\rangle\to\langle\omega'\rangle$ and some low-weight irreps $\{h\}$ and $\{h'\}$ Here p_{st} and q_{st} are defined by $p_{st}=\omega_s-\omega_t+t-s$ and $q_{st}=\omega_s+\omega_t-s-t$ respectively, and $S(i-j)=1$ or -1 according to whether $i\leq j$ or $i>j$

$\{h\}$	$\{q\}$	$\langle\omega'\rangle$	R
$\{\omega+\Delta^{(2)}(j,k)\}$	$\{-1\}$	$\langle\omega-\Delta^{(1)}(i)\rangle$	$(-1)^{i-j}S(i-j)\left[\frac{q_{jk}+1+\delta_{jk}}{q_{ik}}\left(\prod_{s\neq i}\frac{p_{js}+1+\delta_{jk}}{p_{is}}\right)\times\left(\prod_{s\neq j}\frac{p_{is}-1-\delta_{sk}}{p_{js}+\delta_{jk}-\delta_{sk}}\right)\right]^{1/2}$
$\{\omega+\Delta^{(2)}(i,j)\}$	$\{-1\}$	$\langle\omega+\Delta^{(1)}(i)\rangle$	$(-1)^{j-i}\left[\frac{1+\delta_{ij}}{q_{ij}+1+\delta_{ij}}\left(\prod_{s\neq i}\frac{p_{is}-\delta_{sj}}{p_{is}}\right)\times\left(\prod_{s\neq j}\frac{p_{js}+1+\delta_{ji}-\delta_{si}}{p_{js}+\delta_{ji}-\delta_{si}}\right)\right]^{1/2}$
$\{\omega\}$	$\{1\}$	$\langle\omega-\Delta^{(1)}(i)\rangle$	$(-1)^{j-i}\left[\frac{1+\delta_{ij}}{q_{ij}}\left(\prod_{s\neq i}\frac{p_{is}-1-\delta_{sj}}{p_{is}}\right)\right]^{1/2}$
$\{\omega+\Delta^{(2)}(i,k)\}$	$\{1\}$	$\langle\omega+\Delta^{(1)}(i)\rangle$	$S(i-j)\left[\frac{q_{jk}+1+\delta_{ij}+\delta_{jk}+\delta_{ki}}{q_{ik}+1+\delta_{ik}}\times\left(\prod_{s\neq i}\frac{p_{js}+1+\delta_{ij}+\delta_{jk}}{p_{is}}\right)\times\left(\prod_{s\neq j}\frac{p_{is}-\delta_{si}-\delta_{sk}}{p_{js}+1+\delta_{ij}+\delta_{jk}-\delta_{si}-\delta_{sk}}\right)\right]^{1/2}$

Table 3 Ratios R of $\mathrm{sp}(4,\mathbb{R}) \supset \mathrm{u}(2)$ reduced Wigner coefficients for the coupling $\langle \omega+\frac{1}{2}, \omega-\frac{1}{2}\rangle \times (1) \to \langle \omega'+1, \omega'-1\rangle$ in the case where $h_2'-\omega'$ is even

$\{q_1q_2\}$	$\langle \omega'+1, \omega'-1\rangle$	$\{h_1'h_2'\}$	R
$\{-1\}$	$\langle \omega+\frac{1}{2}, \omega-\frac{3}{2}\rangle$	$\{h_1-1, h_2\}$	$-\frac{1}{2}\left[\frac{(h_1-h_2-1)(2h_1+2\omega-3)}{(h_1-h_2)(2\omega-3)}\right]^{1/2}$
$\{-1\}$	$\langle \omega+\frac{1}{2}, \omega-\frac{3}{2}\rangle$	$\{h_1, h_2-1\}$	$\frac{1}{2}\left[\frac{(h_1-h_2+3)(2h_2+2\omega-5)}{(h_1-h_2+2)(2\omega-3)}\right]^{1/2}$
$\{-1\}$	$\langle \omega+\frac{3}{2}, \omega-\frac{1}{2}\rangle$	$\{h_1-1, h_2\}$	$\frac{1}{2}\left[\frac{(h_1-h_2-1)(2\omega-1)(2h_1-2\omega+3)}{(h_1-h_2)(2\omega-2)(2\omega+1)}\right]^{1/2}$
$\{-1\}$	$\langle \omega+\frac{3}{2}, \omega-\frac{1}{2}\rangle$	$\{h_1, h_2-1\}$	$\frac{1}{2}\left[\frac{(h_1-h_2+3)(2\omega-1)(2h_2-2\omega+1)}{(h_1-h_2+2)(2\omega-2)(2\omega+1)}\right]^{1/2}$
$\{1\}$	$\langle \omega+\frac{1}{2}, \omega-\frac{3}{2}\rangle$	$\{h_1, h_2+1\}$	$-\frac{1}{2}\left[\frac{(h_1-h_2-1)(2h_2-2\omega+3)}{(h_1-h_2)(2\omega-3)}\right]^{1/2}$
$\{1\}$	$\langle \omega+\frac{1}{2}, \omega-\frac{3}{2}\rangle$	$\{h_1+1, h_2\}$	$-\frac{1}{2}\left[\frac{(h_1-h_2+3)(2h_1-2\omega+5)}{(h_1-h_2+2)(2\omega-3)}\right]^{1/2}$
$\{1\}$	$\langle \omega+\frac{3}{2}, \omega-\frac{1}{2}\rangle$	$\{h_1, h_2+1\}$	$-\frac{1}{2}\left[\frac{(h_1-h_2-1)(2\omega-1)(2h_2+2\omega-3)}{(h_1-h_2)(2\omega-2)(2\omega+1)}\right]^{1/2}$
$\{1\}$	$\langle \omega+\frac{3}{2}, \omega-\frac{1}{2}\rangle$	$\{h_1+1, h_2\}$	$\frac{1}{2}\left[\frac{(h_1-h_2+3)(2\omega-1)(2h_1+2\omega-1)}{(h_1-h_2+2)(2\omega-2)(2\omega+1)}\right]^{1/2}$

The $\mathrm{sp}(2N,\mathbb{R}) \supset \mathrm{u}(N)$ reduced Wigner coefficients under consideration have therefore been expressed in terms of the K-matrix, the known reduced matrix elements of $\mathbf{z}$ and some $\mathrm{u}(N)$ $6j$- or $9j$-type recoupling coefficients In some important cases, they can be given in very specific analytic form by taking the results of the appendix for the K-matrix into account. As examples, the ratios $R_{r'r}$, defined in (4.2), are listed in tables 1–4 for the following four cases respectively:

N and $\{h\}$ arbitrary, $\langle\omega\rangle = \langle\dot{\omega}\rangle$

N and $\langle\omega\rangle$ arbitrary, $\{h\} = \{\omega\}$ or $\{\omega + \Delta^{(2)}(p,q)\}$, $\{h'\} = \{\omega'\}$ or $\{\omega' + \Delta^{(2)}(s,t)\}$

$N=2$, $\langle\omega_1\omega_2\rangle = \langle\omega+\frac{1}{2}, \omega-\frac{1}{2}\rangle$, $\langle\omega_1'\omega_2'\rangle = \langle\omega'+1, \omega'-1\rangle$, $h_2'-\omega'$ even

$N=2$, $\langle\omega_1\omega_2\rangle = \langle\omega+\frac{1}{2}, \omega-\frac{1}{2}\rangle$, $\langle\omega_1'\omega_2'\rangle = \langle\omega'+1, \omega'-1\rangle$, $h_2'-\omega'$ odd

(note that here a dot over a symbol implies that this symbol is repeated as often as necessary, for instance $\langle\dot{\omega}\rangle = \langle\omega^N\rangle$). The former three cases are multiplicity-free, whereas the latter corresponds to multiplicity-free $\{h\}$ and multiplicity-two $\{h'\}$ irreps. In particular, it is clear that from tables 1, 2 and 4, all the $\mathrm{sp}(4,\mathbb{R}) \supset \mathrm{u}(2)$ reduced Wigner coefficients relating the irreps $\langle\omega\omega\rangle$, $\langle\omega+\frac{1}{2}, \omega-\frac{1}{2}\rangle$, and $\langle\omega+1, \omega-1\rangle$ are explicitly known.

Appendix. The K-matrix elements

The $(KK^\dagger)$-matrix elements can be evaluated from $KK^\dagger(\{\omega\},\{\omega\}) = 1$ through recursion relationships, which follow from (2 13) by taking matrix elements between two

Table 4. Ratios R_r of $\mathrm{sp}(4,\mathbb{R})\supset \mathrm{u}(2)$ reduced Wigner coefficients for the coupling $\langle\omega+\frac{1}{2},\omega-\frac{1}{2}\rangle\times\langle 1\rangle\to\langle\omega'+1,\omega'-1\rangle$ in the case where $h_2'-\omega'$ is odd. The u(2) irrep $\{\nu_1\nu_2\}$ is $\{h_1-\omega+\frac{1}{2},h_2-\omega-\frac{1}{2}\}$ or $\{h_1-\omega-\frac{1}{2},h_2-\omega+\frac{1}{2}\}$ according to whether $h_2-\omega-\frac{1}{2}$ is even or odd

$\{q_1q_2\}$	$\langle\omega'+1,\omega'-1\rangle$	$\{h_1'h_2'\}$	R_r
$\{-1\}$	$\langle\omega+\frac{1}{2},\omega-\frac{3}{2}\rangle$	$\{h_1-1,h_2\}$	$-(K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{3}{2}\},\{h_1-1,h_2\}))_{r,\{\nu_1\nu_2\}}$ $\times K(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{-1\}$	$\langle\omega+\frac{1}{2},\omega-\frac{3}{2}\rangle$	$\{h_1,h_2-1\}$	$(K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{3}{2}\},\{h_1,h_2-1\}))_{r,\{\nu_1\nu_2\}}$ $\times K(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{-1\}$	$\langle\omega+\frac{3}{2},\omega-\frac{1}{2}\rangle$	$\{h_1-1,h_2\}$	$(h_1-h_2)^{-1}\{[(h_1-h_2-1)(h_1-h_2+1)(h_1-\omega+\frac{1}{2})]^{1/2}$ $\times(K(\{\omega+\frac{3}{2},\omega-\frac{1}{2}\},\{h_1-1,h_2\}))_{\{\nu_1-2,\nu_2\},r}$ $+(h_2-\omega+\frac{1}{2})^{1/2}(K(\{\omega+\frac{3}{2},\omega-\frac{1}{2}\},\{h_1-1,h_2\}))_{\{\nu_1,\nu_2-2\},r}\}$ $\times K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{-1\}$	$\langle\omega+\frac{3}{2},\omega-\frac{1}{2}\rangle$	$\{h_1,h_2-1\}$	$(h_1-h_2+2)^{-1}\{[(h_1-h_2+1)(h_1-h_2+3)(h_2-\omega-\frac{1}{2})]^{1/2}$ $\times(K(\{\omega+\frac{3}{2},\omega-\frac{1}{2}\},\{h_1,h_2-1\}))_{\{\nu_1,\nu_2-2\},r}$ $-(h_1-\omega+\frac{3}{2})^{1/2}(K(\{\omega+\frac{3}{2},\omega-\frac{1}{2}\},\{h_1,h_2-1\}))_{\{\nu_1-2,\nu_2\},r}\}$ $\times K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{1\}$	$\langle\omega+\frac{1}{2},\omega-\frac{3}{2}\rangle$	$\{h_1,h_2+1\}$	$-(h_1-h_2)^{-1}\{[(h_1-h_2-1)(h_1-h_2+1)(h_2-\omega+\frac{5}{2})]^{1/2}$ $\times(K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{3}{2}\},\{h_1,h_2+1\}))_{r,\{\nu_1,\nu_2+2\}}$ $+(h_1-\omega+\frac{5}{2})^{1/2}(K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{3}{2}\},\{h_1,h_2+1\}))_{r,\{\nu_1+2,\nu_2\}}\}$ $\times K(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{1\}$	$\langle\omega+\frac{1}{2},\omega-\frac{3}{2}\rangle$	$\{h_1+1,h_2\}$	$(h_1-h_2+2)^{-1}\{-[(h_1-h_2+1)(h_1-h_2+3)(h_1-\omega+\frac{7}{2})]^{1/2}$ $\times(K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{3}{2}\},\{h_1+1,h_2\}))_{r,\{\nu_1+2,\nu_2\}}$ $+(h_2-\omega+\frac{3}{2})^{1/2}(K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{3}{2}\},\{h_1+1,h_2\}))_{r,\{\nu_1,\nu_2+2\}}\}$ $\times K(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{1\}$	$\langle\omega+\frac{3}{2},\omega-\frac{1}{2}\rangle$	$\{h_1,h_2+1\}$	$-(K(\{\omega+\frac{3}{2},\omega-\frac{1}{2}\},\{h_1,h_2+1\}))_{\{\nu_1\nu_2\},r}$ $\times K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$
$\{1\}$	$\langle\omega+\frac{3}{2},\omega-\frac{1}{2}\rangle$	$\{h_1+1,h_2\}$	$(K(\{\omega+\frac{3}{2},\omega-\frac{1}{2}\},\{h_1+1,h_2\}))_{\{\nu_1\nu_2\},r}$ $\times K^{-1}(\{\omega+\frac{1}{2},\omega-\frac{1}{2}\},\{h_1h_2\})$

z-space orthonormal basis states (2.11). Such recursion relations can be written as (Rowe 1984, Deenen and Quesne 1985, Hecht 1987)

$$\sum_{\{\bar\nu\}\bar\rho}[\Lambda(\{\nu'\},\{h'\})-\Lambda(\{\bar\nu\},\{h\})](\langle\omega\rangle\{\nu'\}\rho'\{h'\}\|z\|\langle\omega\rangle\{\bar\nu\}\bar\rho\{h\})$$
$$\times(KK^{\dagger}(\{\omega\},\{h\}))_{\{\bar\nu\}\bar\rho,\{\nu\}\rho}$$
$$=\sum_{\{\bar\nu'\}\bar\rho'}(KK^{\dagger}(\{\omega\},\{h'\}))_{\{\nu'\}\rho',\{\bar\nu'\}\bar\rho'}(\langle\omega\rangle\{\bar\nu'\}\bar\rho'\{h'\}\|z\|\langle\omega\rangle\{\nu\}\rho\{h\}) \qquad \text{(A1)}$$

where the u(N) reduced matrix elements of **z** are given in (3.22) and (3.23), and

$$\Lambda(\{\nu\},\{h\})=\tfrac{1}{2}\sum_i h_i(h_i+N-2i+1)-\tfrac{1}{4}\sum_i\nu_i(\nu_i+2N-2i+2) \qquad \text{(A2)}$$

denotes the eigenvalue of the operator Λ, defined in (2.10), corresponding to the state $|\langle\omega\rangle\{\nu\}\rho\{h\}\chi\rangle$.

For one-dimensional $KK^{\dagger}(\{\omega\},\{h\})$ submatrices, $\{\nu\}$ is entirely determined by $\{h\}$, ρ is unnecessary and one may take $K^{\dagger}(\{\omega\},\{h\})=K(\{\omega\},\{h\})$. Whenever both

$KK^\dagger(\{\omega\},\{h\})$ and $KK^\dagger(\{\omega'\},\{h'\})$ are one-dimensional, equation (A1) simply becomes

$$K^2(\{\omega\},\{h'\}) = [\Lambda(\{\nu'\},\{h'\}) - \Lambda(\{\nu\},\{h\})]K^2(\{\omega\},\{h\}) \tag{A3}$$

for $\{h'\}$ contained in $\{h\}\times\{2\}$, and $\{\nu'\}$ in $\{\nu\}\times\{2\}$. Denoting $\{\nu'\}$ and $\{h'\}$ by $\{\nu+2\Delta^{(1)}(k)\}$ and $\{h+\Delta^{(2)}(i,j)\}$ respectively, we obtain from (A2) and (A3)

$$K^2(\{\omega\},\{h+\Delta^{(2)}(i,j)\}) = (h_i + h_j - \nu_k + k - i - j + \delta_{ij})K^2(\{\omega\},\{h\}) \tag{A4}$$

In such a case, $K(\{\omega\},\{h\})$ is defined as the positive square root of $K^2(\{\omega\},\{h\})$ For higher-dimensional $KK^\dagger(\{\omega\},\{h\})$ submatrices, one has to solve (A1). Then the $K(\{\omega\},\{h\})$ submatrices are determined by diagonalizing $KK^\dagger(\{\omega\},\{h\})$ and applying (2.15) and (2.17).

We give below some examples of results for one- and two-dimensional submatrices.

A1. The case of $\{h\}=\{\omega+\Delta^{(2)}(i,j)\}$

This is an example of application of (A4) with $\{\nu\}=\{0\}$ and $\{\nu'\}=\{2\}$ The result is

$$K^2(\{\omega\},\{\omega+\Delta^{(2)}(i,j)\}) = \omega_i + \omega_j - i - j + 1 + \delta_{ij}. \tag{A5}$$

A2. The case of $\langle\omega\rangle$

All states of $\langle\omega\rangle$ are multiplicity-free with $\{h\}$ given by $h_i = \omega + \nu_i$, $i = 1, \ldots, N$. Application of (A4) leads to (Deenen and Quesne 1982)

$$K^2(\{\dot\omega\},\{h\}) = \prod_i \frac{(h_i + \omega - i - 1)!!}{(2\omega - i - 1)!!} \tag{A6}$$

A3. The case of $\langle\omega+1\ \dot\omega\rangle$

All states of $\langle\omega+1\ \omega\rangle$ are multiplicity-free with $\{h\}$ given by $h_i = \omega + \nu_i + \delta_{im}$, $i = 1, \ldots, N$, for some $m \in \{1, \ldots, N\}$. Direct application of (A4) again leads to

$$K^2(\{\omega+1\ \omega\},\{h\}) = \prod_i \frac{(h_i + \omega - i - 1 + \delta_{im})!!}{(2\omega - i - 1 + 2\delta_{i1})!!} \tag{A7}$$

A4. The case of $\langle\dot\omega\ \omega-1\rangle$

All states of $\langle\dot\omega\ \omega-1\rangle$ are multiplicity-free with $\{h\}$ given by $h_i = \omega + \nu_i - \delta_{im}$, $i = 1, \ldots, N$, for some $m \in \{1, \ldots, N\}$. From (A4), one obtains

$$K^2(\{\dot\omega\ \omega-1\},\{h\}) = \prod_i \frac{(h_i + \omega - i - 1 - \delta_{im})!!}{(2\omega - i - 1 - 2\delta_{iN})!!}. \tag{A8}$$

A5. The case of $\langle\omega+\frac{1}{2}, \omega-\frac{1}{2}\rangle$ for sp(4, $\mathbb{R}$)

This is a special case of both cases 3 and 4. Denoting h_i by $h_i = \omega + \nu_i + \frac{1}{2} - \delta_{im}$, $i = 1, 2$, $m \in \{1, 2\}$, one has

$$K^2(\{\omega+\tfrac{1}{2}, \omega-\tfrac{1}{2}\},\{h_1 h_2\}) = \frac{(h_1 + \omega - \frac{3}{2} - \delta_{m1})!!(h_2 + \omega - \frac{5}{2} - \delta_{m2})!!}{(2\omega-1)!!(2\omega-4)!!}. \tag{A9}$$

A6. The case of $\langle \omega+1, \omega-1\rangle$ *and* $h_2-\omega$ *even for* sp(4, ℝ)

All states of $\langle \omega+1, \omega-1\rangle$ with $h_2-\omega$ even are multiplicity-free with $\{\nu_1\nu_2\}$ given by $\nu_i = h_i - \omega$, $i=1,2$ Application of (A4) gives (Castaños *et al* 1985)

$$K^2(\{\omega+1, \omega-1\}, \{h_1h_2\}) = (2\omega-2)\frac{(h_1+\omega-2)!!(h_2+\omega-3)!!}{(2\omega)!!(2\omega-3)!!}. \tag{A10}$$

A7. The case of $\langle \omega+1, \omega-1\rangle$ *and* $h_2-\omega$ *odd for* sp(4, ℝ)

The states of $\langle \omega+1, \omega-1\rangle$ with $h_2-\omega$ odd have multiplicity two and correspond to $\{\nu_1\nu_2\} = \{h_1-\omega-1, h_2-\omega+1\}$ or $\{h_1-\omega+1, h_2-\omega-1\}$. The matrix elements of $KK^\dagger(\{\omega+1, \omega-1\}, \{h_1h_2\})$, obtained by solving (A1), can be written as (Castaños *et al* 1985)

$$(KK^\dagger(\{\omega+1, \omega-1\}, \{h_1h_2\}))_{\{\nu_1'\nu_2'\},\{\nu_1\nu_2\}} = \frac{(h_1+\omega-3)!!(h_2+\omega-4)!!}{(2\omega)!!(2\omega-3)!!} A_{\{\nu_1'\nu_2'\}\{\nu_1\nu_2\}} \tag{A11}$$

where

$$A_{\{h_1-\omega-1,h_2-\omega+1\},\{h_1-\omega-1,h_2-\omega+1\}} = (2\omega-2)(h_1+\omega-2) - \frac{(h_1-h_2)(h_1-\omega+2)}{h_1-h_2+1}$$

$$A_{\{h_1-\omega-1,h_2-\omega+1\},\{h_1-\omega+1,h_2-\omega-1\}} = -\frac{[(h_1-h_2)(h_1-h_2+2)(h_1-\omega+2)(h_2-\omega+1)]^{1/2}}{h_1-h_2+1}$$

$$A_{\{h_1-\omega+1,h_2-\omega-1\},\{h_1-\omega+1,h_2-\omega-1\}} = (2\omega-2)(h_2+\omega-3) - \frac{(h_1-h_2+2)(h_2-\omega+1)}{h_1-h_2+1}. \tag{A12}$$

The eigenvalues of the 2×2 matrix $KK^\dagger(\{\omega+1, \omega-1\}, \{h_1h_2\})$ are given by

$$d_{1,2} = \frac{(h_1+\omega-3)!!(h_2+\omega-4)!!}{(2\omega)!!(2\omega-3)!!}\tfrac{1}{2}[(2\omega-3)(h_1+h_2)+4(\omega-1)(\omega-2)\pm\Delta] \tag{A13}$$

where d_1 (d_2) corresponds to the + (−) sign, and

$$\Delta = [4(\omega-1)(\omega-2)(h_1-h_2)(h_1-h_2+2)+(h_1+h_2)^2]^{1/2}. \tag{A14}$$

The (real) unitary matrix U converting $KK^\dagger(\{\omega+1, \omega-1\}, \{h_1h_2\})$ to diagonal form can be written as

$$U = \begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{A15}$$

where

$$\sin\phi = \left[\frac{-(2\omega-3)(h_1-h_2)(h_1-h_2+2)-h_1-h_2+(h_1-h_2+1)\Delta}{2(h_1-h_2+1)\Delta}\right]^{1/2} \tag{A16}$$

$$\cos\phi = \left[\frac{2(h_1-h_2)(h_1-h_2+2)(h_1-\omega+2)(h_2-\omega+1)}{(h_1-h_2+1)\Delta[-(2\omega-3)(h_1-h_2)(h_1-h_2+2)-h_1-h_2+(h_1-h_2+1)\Delta]}\right]^{1/2}.$$

The 2×2 matrices $K(\{\omega+1, \omega-1\}, \{h_1h_2\})$ and $K^{-1}(\{\omega+1, \omega-1\}, \{h_1h_2\})$ are finally obtained by applying (2.17).

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